Handout for Presentation on L^p-Improving Measures in Abstract Harmonic Analysis

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Short Bio

I completed a Honors Bachelor of Science in mathematics (specialist program, pure math stream) at the University of Toronto Scarborough. This was actually part of the Concurrent Teacher Education Program, which I am now in my last year of at the Ontario Institute for Studies in Education for teaching at the secondary level in Ontario (mathematics and computer studies). This presentation was part of a larger report done for a reading course in Abstract Harmonic Analysis. My future plans are either to teach mathematics and computer studies at the secondary level, or to continue my studies in mathematics with a masters degree.

L^p-Improving Measures

The following is part of a report done for a reading course with Dr. Raymond Grinnell which was completely based on [1]. L^p-improving measures is a specific and important area within abstract harmonic analysis, a particular kind of analysis in mathematics. To introduce this area some preliminaries are needed. Let G be an infinite compact abelian group. This means a few things: G is a group with an operation $\cdot$ which is commutative, it has infinitely many elements, and is topologically compact. For example, G could be the *circle group* $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ where $|z|$ is the complex modulus. Although this is a very important example, here G will always be arbitrary.

The next thing needed is a *Haar measure* $\lambda : \mathcal{B}(G) \to [0, \infty]$ on G where $\mathcal{B}(G)$ is the Borel σ-algebra of G. λ is defined to be a Haar measure by satisfying the following: (1) λ is a positive measure, (2) λ is regular, (3) for any nonempty open set U, $\lambda(U) > 0$, (4) there exists a nonempty open set U such that $\lambda(U) < \infty$, (5) for any compact subset K, $\lambda(K) < \infty$, and (6) λ is *left-invariant*, i.e. $\forall b \in G$, $\forall A \in \mathcal{B}(G)$, $\lambda(bA) = \lambda(A)$. Moreover λ must be *normalized*, i.e. $\lambda(G) = 1$.

This allows the construction of the set $L^p(G)$ for a given $1 \le p \le \infty$. For $p < \infty$, $L^p(G)$ is the collection of functions $f : G \to \mathbb{C}$ satisfying

$$\|f\|_p = \left(\int_G |f|^p \, d\lambda \right)^{1/p} < \infty$$

In fact $L^p(G)$ is a normed vector space with the above norm. A very useful fact is that because G is assumed to be compact, it follows that for $1 < p < q < \infty$, $L^q(G) \subsetneq L^p(G)$. And from these functions an essential operation can be defined as follows. Denote by $M(G)$ the collection of all complex regular measures on G, so that $M(G)$ includes λ. Then for $\mu \in M(G)$ and $f \in L^p(G)$, define the *convolution* of μ and f to be the function

$$\mu * f : G \to \mathbb{C},\ x \mapsto \int_G f\left(s^{-1}x\right) d\mu(s)$$

It is a fact that $f \in L^p(G)$ implies $\mu * f \in L^p(G)$. Moreover, if $1 < p < q < \infty$ and

$$\mu * L^p(G) = \{\mu * f : f \in L^p(G)\} \subseteq L^q(G)$$

then μ is said to be L^p*-improving*. This is interesting because as noted earlier $p < q$ implies $L^q(G) \subsetneq L^p(G)$, so μ "transforms" $L^p(G)$ to "reverse" the containment. Many such measures exist, including λ itself!

It is important to notice that in defining an L^p-improving measure a particular p is needed. However the following theorem shows that the p itself is not special provided at least one is shown to exist.

Theorem Suppose $\mu \in M(G)$ is L^p-improving for some $1 < p < \infty$. Then for any $1 < r < \infty$, μ is also L^r-improving.

Proof. Since μ is L^p-improving, therefore there is some $p < q < \infty$ such that $\mu * L^p(G) \subseteq L^q(G)$. It turns out that this is equivalent to the linear transformation

$$T_\mu : L^p(G) \to L^q(G),\ f \mapsto \mu * f$$

being bounded. Moreover it is a fact that T_μ as a transformation $L^\infty(G) \to L^\infty(G)$ (with the same mapping) is also bounded. A theorem by the name of the Riesz-Thorin (Convexity) theorem then states that $\forall \theta \in (0,1)$, it is also true that $T_\mu : L^{p_\theta}(G) \to L^{q_\theta}(G)$ is bounded where $p_\theta = p/\theta$ and $q_\theta = q/\theta$. Choosing $\theta = p/r$ results in $p_\theta = r$. Then since $p < q$ implies $r = p_\theta < q_\theta$, the equivalence noted at the beginning of this proof then shows that μ is L^r-improving, provided $\theta = p/r < 1$. However the latter only holds if $r \in (p, \infty)$. To prove the remainder of the theorem, namely for $r \in (1, p)$, a similar argument is done using the boundedness of $T_\mu : L^1(G) \to L^1(G)$. ■

Consequently more generally a measure $\mu \in M(G)$ is *Lebesgue-improving* if it is L^p-improving for some p. From here more elementary properties of L^p-improving measures can be proved, such as that linear combinations and convolution products of Lebesgue-improving measures results in Lebesgue-improving measures, and additional examples and non-examples. Moreover there are additionally sophisticated ways of characterizing L^p-improving measures such as in terms of "size" (e.g. in terms of the *Fourier transform*) and another area of study in abstract harmonic analysis called lambda-p ($\Lambda(p)$) sets.

References

[1] R. J. Grinnell, *Lorentz-Improving Measures on Compact Abelian Groups*. Ph.D. dissertation, Queen's University, 1991.