Exploring Menelaus' Theorem in Hilbert Geometry

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Bio

I am a mathematics major in my final year of studies. I am also enrolled in the Concurrent Teacher Education Program and will be attending the Ontario Institute for Studies in Education next year for completing my Bachelor of Education degree. In the future, I plan to teach mathematics at the secondary level.

Introduction

Menelaus was a Greek mathematician born around 70 AD in Alexandria. He explored geometry in depth, and his most notable work titled *Sphaerica* detailed spherical triangles and their applications to astronomy [3]. Perhaps one of his most well known contributions to mathematics, Menelaus' theorem states that points lying on three sides of a triangle are collinear if and only if the ratio of the products of the non-adjacent sides is 1. As evident, Menelaus' theorem is closely related to Ceva's theorem. While the former demonstrates when certain points on a triangle are collinear, the latter explores when certain line segments of a triangle are concurrent. This property of similarity between the theorems is called duality [3]. Moreover, for both theorems, the converses hold true as well.

Menelaus' Theorem in Hyperbolic Geometry

Given the different kinds of geometries we encounter in mathematics, there are several versions of Menelaus' theorems that are used. When we work with this theorem in Euclidean geometry, we use signed ratios. Although we can define similar ratios in hyperbolic geometry, they won't be very useful quantities. Hence, we use another ratio, namely the hyperbolic ratio:

Definition 1 (Hyperbolic Ratio). *Let A, B and X be three distinct points on a hyperbolic line. Their hyperbolic ratio is*

$$h(A,X,B)=\begin{cases} \dfrac{\sin h\big(d(A,X)\big)}{\sin h\big(d(X,B)\big)}, & \text{if X is between A and B} \\ -\dfrac{\sin h\big(d(A,X)\big)}{\sin h\big(d(X,B)\big)}, & \text{for all other cases} \end{cases}$$

where $\sin h\big(d(A,X)\big)$ *is the hyperbolic sine function.*

As shown in the above, it is important to note that the value of h(A,X,B) determines the position of point X relative to the positions of points A and B. We now define Menelaus' theorem in hyperbolic geometry using the hyperbolic ratio:

Theorem 1 (Menelaus in Hyperbolic Geometry). *Let ABC be a hyperbolic triangle. Let L be a hyperbolic line that does not pass through any vertex of Δ ABC but meets BC at Q, AC at R, and AB at P. Then, the absolute value of the product of their hyperbolic ratios is 1. That is,*

$$|h(A,P,B)\ h(B,Q,C)\ h(C,R,A)| = 1$$

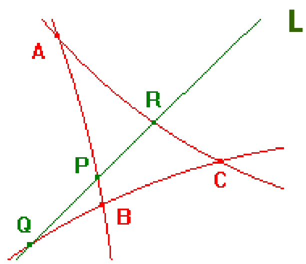

Figure 1. Menelaus' theorem in hyperbolic geometry [4].

Proof. Since we want to show that the absolute value of the product of the hyperbolic ratios is 1, we can label the vertices of Δ ABC in any order and it won't impact our result. We apply the hyperbolic sine rule to Δ APR and get the following:

$$\frac{\sin h\ (d(A,R))}{\sin(<APR)} = \frac{\sin h\ (d(A,P))}{\sin(<ARP)}$$

Likewise, from Δ BPQ and Δ CRQ, we get the following:

$$\frac{\sin h\ (d(B,Q))}{\sin(<BPQ)} = \frac{\sin h\ (d(B,P))}{\sin(<BQP)}$$

$$\frac{\sin h\ (d(C,R))}{\sin(<CQR)} = \frac{\sin h\ (d(C,Q))}{\sin(<CRQ)}$$

From figure 1, we see the following relations:

$<APR = <BPQ$, $<BQP = <CQR$, $<ARP = \pi - <CRQ$, and hence $\sin(<APR) = \sin(<BPQ)$, $\sin(<BQP) = \sin(<CQR)$, and $\sin(<ARP) = \sin(<CRQ)$.

The above equations lead us to the required product, which is:

$$|h(A, P, B)\ h(B, Q, C)\ h(C, R, A)| = 1$$

Note that the above proof is valid for one case only. There are other cases possible, all of which can be proven using a similar method. Q. E. D.

Hilbert Geometry

Serving as a connection between modern and historical geometries, Menelaus' theorem has certain applications in higher-level mathematics. More specifically, the hyperbolic version of this theorem has been used to explore new approaches in geometry. One such application is Hilbert geometry, which was introduced by David Hilbert in 1899 and includes natural generalizations of hyperbolic geometry [2].

Let's consider two points A and B in $\mathbb{R}^n$ and denote the line passing through them by AB as well as the open segment with end points A and B by $\overline{AB}$ [2]. We now define the cross ratio in Hilbert geometry:

Definition 2 (Cross Ratio). *Let A and B be distinct points in $\mathbb{R}^n$, with points $X, Y \in AB$ such that they can be expressed using the linear combinations $X = \lambda_1 A + \mu_1 B$ and $Y = \lambda_2 A + \mu_2 B$, where λ_1, λ_2, μ_1, μ_2 are real numbers. Then, the cross ratio of A, B, X and Y is*

$$(A, B; X, Y) = \frac{\mu_1 \lambda_2}{\lambda_1 \mu_2}, \text{ where } \lambda_1 \mu_2 \neq 0$$

Then, we define Hilbert geometry:

Definition 3 (Hilbert Geometry). *Let $\mathcal{H}$ be an open, strictly convex set in $\mathbb{R}^n$, $n \geq 2$, with boundary $\partial\mathcal{H}$. The Hilbert metric on $\mathcal{H}$ is the function $d_{\mathcal{H}} : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{R}$ such that*

$$d_{\mathcal{H}}(X, Y) = \begin{cases} 0, & if\ X = Y \\ \frac{1}{2}\left|\ln |A, B;\ X, Y|\right|, & if\ X \neq Y\ and\ \overline{AB} = \mathcal{H}\ \cap XY \end{cases}$$

The pair $(\mathcal{H}, d_{\mathcal{H}})$ is called the Hilbert geometry in $\mathcal{H}$.

It is shown that a Hilbert geometry in which Menelaus' Theorem holds true is hyperbolic. The reason for this is that such a geometry turns out to be a solid with all sections that are ellipses, i.e. an ellipsoid. However, we find that $\mathcal{H}$ is an ellipsoid if and only if it is hyperbolic [2].

References

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[4] Proof of Menelaus' Theorem http://www.maths.gla.ac.uk/wws/cabripages/hyperbolic/pmenelaus.html